Section 13.3 – 13.5 Worksheet Solutions

Section 13.3 Additional Exercises

1. $v \cdot e = \|v\| \cdot \|e\| \cos\theta = 3 \cdot 1 \cdot \cos(2\pi/3)$

$= 3 \cdot (-1/2) = \boxed{-3/2}$

2. We will use the fact that $\|e+f\| = 3/2$ to find the value of $e \cdot f$.

We have: $\left(\frac{3}{2}\right)^2 = \|e+f\|^2 = (e+f)\cdot(e+f)$

$= e \cdot e + 2e \cdot f + f \cdot f$

$= \|e\|^2 + 2e \cdot f + \|f\|^2$

$= 2 + 2e \cdot f$. [since $\|e\|^2 = \|f\|^2 = 1$ b/c e and f are unit vectors]

Solving for $e \cdot f$:

$\frac{9}{4} = 2 + 2e \cdot f \iff \frac{\frac{9}{4} - 2}{2} = e \cdot f$

$\iff \frac{1}{8} = e \cdot f.$

So, $\|e-f\|^2 = (e-f)\cdot(e-f)$

$= e \cdot e - 2e \cdot f + f \cdot f$

$= \|e\|^2 - 2e \cdot f + \|f\|^2$

$= 2 - 1/4$

$= 7/4 \implies \boxed{\|e-f\| = \sqrt{7}/2}$

Section 13.4 Additional Exercises

1. We say that $\{v, w, u\}$ forms a right-handed system if the direction of u is determined by the right-hand rule:

i.e. When the fingers of your right hand curl from v to w, your thumb points to the same side of the plane spanned by v and w as u. [see p. 671 Figure 3]

(a) no (b) yes (c) yes

(d) no (e) no (f) no

2. $$v \times w = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = (2\cdot 1 - 1)\mathbf{i} - (1-3)\mathbf{j} + (1-6)\mathbf{k}$$

$$= \mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$$

$$= \boxed{\langle 1, 2, -5 \rangle}$$

Section 13.5 Additional Exercises

1. Step 1: Find normal vector to the plane. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in the plane $\mathcal{P}$, so their cross product is normal to $\mathcal{P}$.

$$\overrightarrow{PQ} = \langle 1, -2, 3 \rangle, \quad \overrightarrow{PR} = \langle -1, -2, 6 \rangle$$

↳

$$n = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ -1 & -2 & 6 \end{vmatrix} = (-12+6)\mathbf{i} - (9)\mathbf{j} + (-2-2)\mathbf{k}$$

$$= -6i - 9j - 4k$$

$$= \langle -6, -9, -4 \rangle.$$

Then, P has equation $-6x - 9y - 4z = d$ for some d.

Step 2.) Choose a point on the plane and compute d. Take $Q = (1,1,1)$.

Then, $-6 - 9 - 4 = -19$. So, an equation for P is

$$-6x - 9y - 4z = -19 \quad \text{or}$$

$$\boxed{6x + 9y + 4z = 19}$$

2. We are looking for a plane P that is parallel to $x+y+z=3$. So, it must have the same normal vector (or a multiple of it).

That means P satisfies $x+y+z=d$ for some d. To find d we use that P contains $(4,1,9)$.

So, $4+1+9 = 14 = d$.

An equation for P is $\boxed{x+y+z=14}$

3. From Figure 9 on the worksheet, the angle between two planes is the angle between their normal vectors. A normal vector to the plane $3x+y-4z=2$ is $\langle 3,1,-4\rangle$.

So, the plane we are looking for must have a normal vector $n=\langle n_1,n_2,n_3\rangle$ such that

$\langle n_1,n_2,n_3\rangle \cdot \langle 3,1,-4\rangle = 0$ [b/c an angle of $\pi/2$ means they are perpendicular]

$\Leftrightarrow 3n_1+n_2-4n_3=0$

By inspection, we see that

$n=\langle 1,1,1\rangle$ satisfies this.

So, all planes with equation $x+y+z=d$ for some d, make an angle of $\pi/2$ with the plane $3x+y-4z=2$.

$\boxed{\text{One such plane is } x+y+z=0}$

↳

4. We need to find a plane whose normal vector is perpendicular to the normal vector of both $x+y=3$ and $x+2y-z=4$

$n_1 = \langle 1,1,0 \rangle$ $n_2 = \langle 1,2,-1 \rangle$

This can be done by finding $n_1 \times n_2$.

So, $n = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{vmatrix} = (-1)i - (-1)j + (1)k$
$= -i + j + k$
$= (-1, 1, 1)$

So, planes with equations of the form

$-x+y+z=d$ are perpendicular to

both planes. In particular, the plane

$\boxed{-x+y+z=0 \text{ is perp. to both planes.}}$

5. Let P_1 and P_2 be planes with normal vectors n_1 and n_2 as in the figure

above. If P_1 and P_2 are not parallel, then their intersection is a line $\mathcal{L}$ with direction vector $n = n_1 \times n_2$.
[see figure 9 in worksheet]

↳

In the case of this problem, we can take

$n_1 = \langle 1, -1, -1 \rangle$ and $n_2 = \langle 2, 3, 1 \rangle$.

Then, $n = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 2 & 3 & 1 \end{vmatrix} \begin{aligned} &(-1+3)i - (1+2)j + (3+2)k \\ &= 2i - 3j + 5k \\ &= \langle 2, -3, 5 \rangle. \end{aligned}$

So, $\mathcal{L}$ has parametric equations

$$x = x_0 + 2t, \quad y = y_0 - 3t, \quad z = z_0 + 5t$$

Where $P = (x_0, y_0, z_0)$ is a point in $\mathcal{L}$.

[i.e. P is contained in both planes]

Lets set $z_0 = 2$, then x_0, y_0 must satisfy:

$$\begin{cases} x_0 - y_0 = 3 & (\text{Eq. }1) \\ 2x_0 + 3y_0 = 0 & (\text{Eq. }2) \end{cases}$$

$\Rightarrow x_0 = 3 + y_0$ plugging into Eq. 2 gives

$$2(3 + y_0) + 3y_0 = 0 \iff$$

$$6 + 2y_0 + 3y_0 = 0 \iff$$

$$6 + 5y_0 = 0 \iff$$

$$y_0 = -6/5 \Rightarrow x_0 = 9/5$$

So, $\mathcal{L}$ has parametric equations

$$\boxed{x = 9/5 + 2t, \quad y = -6/5 - 3t, \quad z = 2 + 5t.}$$